Due Tues

3.2 – Norm, Dot Product, and Distance in R^n

Definition 1: If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the **norm** of $\mathbf{v}$ (also called its **length** or **magnitude**) is denoted by $\|\mathbf{v}\|$ [by this author], and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$\vec{u} = (a, b)$ b $\|\vec{u}\| = \sqrt{a^2+b^2}$

a

Theorem 3.2.1 Properties of the norm of a vector

If $\mathbf{v}$ is a vector in R^n and k is any scalar, then:

a) $\|\mathbf{v}\| \geq 0$

b) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

c) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$ ← good practice exercise

The norm of a **unit vector** is 1.

$$\left\|\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

We can obtain a unit vector from a nonzero vector $\mathbf{v}$ by multiplying by the reciprocal of its length. This process is called **normalizing** the vector.

1. Find the norm of $\mathbf{v}$, and a unit vector that is oppositely directed to $\mathbf{v}$.

a. $\mathbf{v} = (2, 2, 2)$

b. $\mathbf{v} = (1, 0, 2, 1, 3)$

b. $\|\vec{v}\| = \sqrt{1+4+1+9} = \sqrt{15}$

$$-\frac{1}{\|\vec{v}\|}\vec{v} = -\frac{1}{\sqrt{15}}(1, 0, 2, 1, 3) = \left(-\frac{1}{\sqrt{15}}, 0, -\frac{2}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}}\right)$$

$(2, -3, 5) = 2\vec{e}_1 - 3\vec{e}_2 + 5\vec{e}_3 = 2(1,0,0) - 3(0,1,0) + 5(0,0,1)$

The **standard unit vectors in R^n** are the standard basis vectors for R^n, $\mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, 0, 0, \dots, 1)$.

Every vector in $\mathbb{R}^n$ can be expressed as a linear combination of these.

$$\vec{v} = (v_1, v_2, \cdots, v_n)$$
$$= v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$$

Definition 2: If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are points in R^n, then we denote the **distance** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

4. Evaluate the given expression with $\mathbf{u} = (2, -2, 3)$, $\mathbf{v} = (1, -3, 4)$, and $\mathbf{w} = (3, 6, -4)$.

a. $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$

b. $\|\mathbf{u} - \mathbf{v}\| \leftarrow$ distance

c. $\|3\mathbf{v}\| - 3\|\mathbf{v}\| = 0$

d. $\|\mathbf{u}\| - \|\mathbf{v}\|$

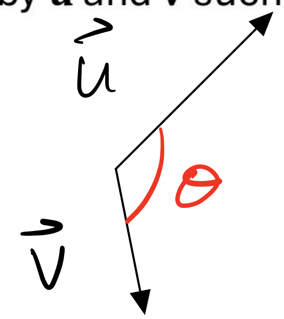
b) $\vec{u} - \vec{v} = (2-1, -2+3, 3-4) = (1, 1, -1)$

$$\|\vec{u} - \vec{v}\| = \sqrt{3}$$

d) $\|\vec{u}\| = \sqrt{4+4+9} = \sqrt{17}$, $\|\vec{v}\| = \sqrt{1+9+16} = \sqrt{26}$

$$\|\vec{u}\| - \|\vec{v}\| = \sqrt{17} - \sqrt{26}$$

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in R^n positioned so that their initial points coincide. The **angle between $\mathbf{u}$ and $\mathbf{v}$** is the angle θ determined by $\mathbf{u}$ and $\mathbf{v}$ such that $0 \leq \theta \leq \pi$.

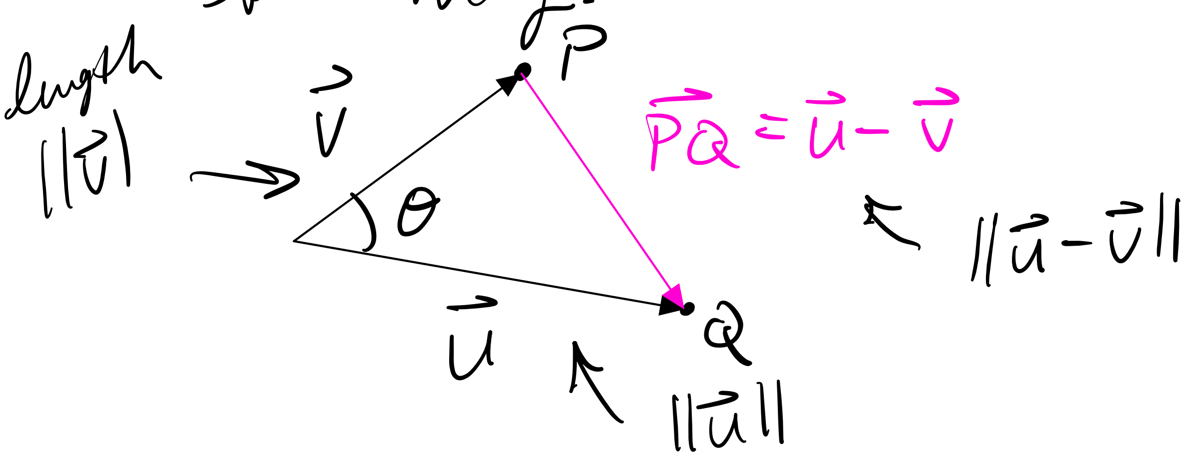


Definition 3: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the **dot product** or **Euclidean inner product** of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

Definition 4: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the **dot product** or **Euclidean inner product** of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by
$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

To connect these two definitions, let $\vec{u}$ & $\vec{v}$ share an initial point in R^3 and let Q & P be their terminal points, respectively.



By the Law of Cosines,

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta = \|\vec{u} - v\|^2$$

$$\Rightarrow \|\vec{u}\|\,\|\vec{v}\|\cos\theta = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right)$$

$$= \frac{1}{2}\Big(u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2$$
$$- (u_1 - v_1)^2 - (u_2 - v_2)^2 - (u_3 - v_3^2)\Big)$$

$$\boxed{\|\vec{u}\|\,\|\vec{v}\|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3 = \vec{u}\cdot\vec{v}}$$

10. Find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{v} \cdot \mathbf{v}$.

a. $\mathbf{u} = (1, 1, -2, 3)$, $\mathbf{v} = (-1, 0, 5, 1)$

b. $\mathbf{u} = (2, -1, 1, 0, -2)$, $\mathbf{v} = (1, 2, 2, 2, 1)$

$$\vec{u}\cdot\vec{v} = 2(1) + (-1)(2) + 1(2) + 0(2) + (-2)(1)$$
$$= 0 \longleftarrow \theta = \pi/2$$

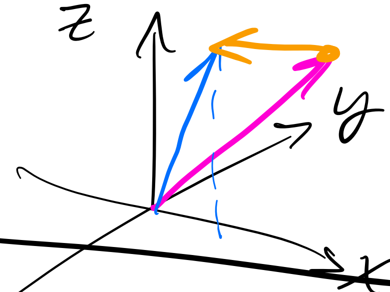
$$\vec{u}\cdot\vec{u} = 2^2 + (-1)^2 + 1^2 + 0^2 + (-2)^2 = 10 = \|\vec{u}\|^2$$

$$\vec{v}\cdot\vec{v} = 1^2 + 2^2 + 2^2 + 2^2 + 1^2 = 14$$

11. Find the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$ and the cosine of the angle between those vectors. State whether that angle is acute, obtuse, or 90°.

a. $\mathbf{u} = (3, 3, 3)$, $\mathbf{v} = (1, 0, 4)$

b. $\mathbf{u} = (0, -2, -1, 1)$, $\mathbf{v} = (-3, 2, 4, 4)$



a) $d(\vec{u}, \vec{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$

$= \sqrt{(3-1)^2 + 3^2 + (3-4)^2} = \sqrt{14}$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \dfrac{3 + 12}{3\sqrt{3}\sqrt{17}} = \dfrac{5}{\sqrt{51}}$

since $\cos\theta > 0$, $0 < \theta < \pi/2$ acute

b) $\vec{u} \cdot \vec{v} < 0 \Rightarrow \theta$ is obtuse

Theorem 3.2.2 Properties of the dot product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and if k is a scalar, then:

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetry property)

b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive property)

c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ (homogeneity property)

d) $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positivity property)

Pf b: Let $\vec{u}=(u_1,u_2,\dots,u_n)$, $\vec{v}=(v_1,v_2,\dots,v_n)$, and $\vec{w}=(w_1,w_2,\dots,w_n)$.

$$\vec{u}\cdot(\vec{v}+\vec{w}) = (u_1,u_2,\dots,u_n)\cdot(v_1+w_1, v_2+w_2,\dots,v_n+w_n)$$

(def of vector addition)

$$= u_1(v_1+w_1)+u_2(v_2+w_2)+\dots+u_n(v_n+w_n)$$

(def of dot product)

$$= u_1v_1+u_1w_1+u_2v_2+u_2w_2+\dots+u_nv_n+u_nw_n$$

(dist. prop. of mult. of real #s)

$$= u_1v_1+u_2v_2+\dots+u_nv_n+u_1w_1+u_2w_2+\dots+u_nw_n$$

(commutative prop. of add. of real #s)

$$= \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$$

(def of dot product)

Theorem 3.2.3 More properties of the dot product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and if k is a scalar, then:

a) $\mathbf{0}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{0}=0$

b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$

c) $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w})=\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{w}$

d) $(\mathbf{u}-\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}-\mathbf{v}\cdot\mathbf{w}$

e) $k(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot(k\mathbf{v})$

— a necessary result for the def of dot product

Theorem 3.2.4 Cauchy-Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \dots u_n)$ and $\mathbf{v} = (v_1, v_2, \dots v_n)$ are vectors in R^n, then $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq (u_1^2 + u_2^2 + \cdots u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots v_n^2)^{1/2}$$

Theorem 3.2.5 Triangle Inequalities

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, then:

a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality for vectors)

b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (triangle inequality for distances)

Pf (a): $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

$$\leq \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$$

$$= (\|\vec{u}\| + \|\vec{v}\|)^2$$

So $\|\vec{u}+\vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2$

Taking square roots yields $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ ✓

Part (b) follows from part (a) and Def 2.